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COMPRESSIBLE STRATIFIED FLOWS OVER A LONG OBSTACLE,(U)

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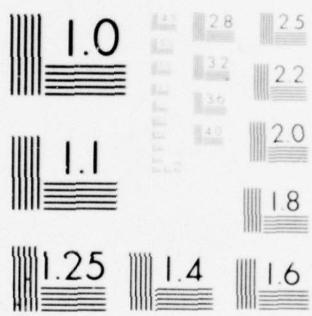

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Compressible Stratified Flows Over a Long Obstacle

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Abstract

Steady two-dimensional stratified shear flows of a compressible fluid over a mountain is studied with the long-wave assumption (hydrostatic balance). For constant entropy profile far upstream, we obtained a close form solution. For general stratification and shear, the speed of certain points in the flow may vanish. We treated these points as the birth-bed of flow fields with closed streamlines (recirculating region). Constants of motion along a streamline are used to construct the outer boundary of these regions. Numerical solutions with data upstream to the Rocky Mountains were constructed with the recirculating region in the flow field.

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1. Introduction

The flow of compressible stratified fluids over an obstacle is the subject of this paper. Assuming a steady unperturbed state far upstream of the obstacle and the existence of either a free or rigid surface as the upper boundary condition (see figure 1), we determine the resulting flow field with or without closed streamlines. We emphasize the introduction of closed streamlines. It is shown that for certain upstream conditions and some obstacle height, the velocity of the fluid vanishes at definite points within the flow field. It is at these points that the regions of closed streamlines originate. For simplicity we restrict ourselves to the case of steady, two dimensional, isentropic, compressible flows in the presence of gravity.

In section 2, using the hydrostatic or long wave approximation we derive a pair of first order, nonlinear, ordinary differential equations which govern the flow in this problem. In the derivation of these equations we obtain three constants of motion along streamlines which prove useful in the analysis of flows with closed streamlines. We also observe that the assumption of hydrostatic balance, along with the condition of either a free or rigid upper boundary, allows us to solve the problem in the following indirect manner: given upstream velocity and entropy profiles and the velocity of the fluid along the upper boundary, find the velocities in the interior of the flow field and the height of the obstacle. In fact, the flow at any downstream station depends only on the conditions far upstream of the obstacle and on the height of the obstacle immediately below.

This allows us to construct a master solution for given upstream conditions which contains all possible steady flow fields corresponding to obstacles of height less than or equal to the critical height, which is the maximum obstacle height allowed by the solution. The corresponding procedure for incompressible flow without closed streamlines was introduced by Lee and Su (1977).

In section 3 we assume that the flow is homentropic (i.e. that entropy is constant throughout the fluid) and show that in this case the system of ordinary differential equations derived in section 2 can be solved analytically. For a constant velocity profile far upstream of the obstacle, we obtain a master solution which contains all possible steady flow fields corresponding to a particular choice of the Froude number. Yih (1965) constructed solutions for homentropic flows under the assumptions that specific energy is constant throughout the fluid and that the dynamic compressibility can be neglected. Yih (1965) also constructed solutions for flows corresponding to slight stratifications in entropy and specific energy assuming again that dynamic compressibility may be neglected. These solutions differ from ours in that Yih did not make the long wave approximation. Consequently his method yields solutions containing lee waves, while our method does not.

In section 4 we allow variable entropy and velocity profiles far upstream of the obstacle. In this case analytic solutions are no longer possible and thus we resort to numerical techniques for the integration of our system of ordinary differential equations. Solutions with upstream conditions corresponding to

flow over the Rocky Mountains are obtained. These solutions typically contain one or more stagnation points in the interior of the flow field. Appropriate use of the three constants of motion along streamlines, obtained in section 2, provide us with three constraining conditions to construct discontinuous solutions containing split streamlines. It should be noted however, that our analysis does not provide any information concerning the flow in the recirculating region. The method used to study the splitting streamlines is closely related to the one used by Margolis and Su (1976) in their analysis of flows with a critical layer.

2. Equations of Motion

The equations governing steady, compressible, isentropic, two dimensional flows in the presence of gravity are the horizontal and vertical momentum equations

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} , \tag{2.1}$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial y} - g, \tag{2.2}$$

the equation of continuity

$$\frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0 , \tag{2.3}$$

and the equation of state

$$p/\rho^{\gamma} = A(s) , \tag{2.4}$$

where s denotes entropy and γ is the ratio of specific heats $\gamma = c_p/c_v$.

From equations (2.1),(2.2),(2.3) and (2.4) it can be shown that

$$A = A(\psi) \tag{2.5}$$

$$H = \frac{1}{2}(u^2+v^2) + gy + \frac{c^2}{\gamma-1} = H(\psi) \tag{2.6}$$

and $$\omega = [\tfrac{1}{2}(u^2+v^2)+gy]\frac{dA}{d\psi} + \frac{A\gamma}{\rho}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \omega(\psi) \tag{2.7}$$

where ψ is the stream function defined by

$\rho u = \frac{\partial \psi}{\partial y}$ and $\rho v = -\frac{\partial \psi}{\partial x}$, and $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \gamma \frac{p}{\rho}$.

Using the hydrostatic relation we replace the vertical momentum equation (2.2) with

$$\frac{\partial p}{\partial y} - \rho g = 0. \tag{2.8}$$

Equations (2.5),(2.6) and (2.7) become

$$A = A(\psi) \tag{2.9}$$

$$H = \frac{u^2}{2} + gy + \frac{c^2}{\gamma-1} = H(\psi) \tag{2.10}$$

and $$\omega = \left(\frac{u^2}{2} + gy\right)\frac{dA}{d\psi} + \frac{A\gamma}{\rho}\frac{\partial u}{\partial y} = \omega(\psi). \tag{2.11}$$

Following Long (1953) we define a new vertical coordinate $Y = Y(\psi) = Y(x,y)$ as the height far upstream of the streamline which passes through the point (x,y) (see figure 2). We then have that the horizontal component of the fluid velocity far upstream is given by $U(Y) = \frac{1}{\rho_o}\frac{\partial \psi}{\partial Y}$, where $\rho_o(Y)$ is the density far upstream. Equation (2.11) now takes the form

$$\left(\frac{u^2}{2} - \frac{U^2}{2}\right) \frac{\partial A}{\partial Y} - A\gamma\left(u \frac{\partial u}{\partial Y} - U \frac{\partial U}{\partial Y}\right) + g(y-Y) \frac{\partial A}{\partial Y} = 0$$

or equivalently

$$\frac{d}{dY} \left[A^{-1/\gamma}\left(\frac{U^2}{2} - \frac{u^2}{2}\right)\right] - g(y-Y) \frac{dA^{-1/\gamma}}{dY} = 0. \tag{2.12}$$

Introducing $\zeta(Y) = y-Y$ we have that

$$\frac{d\zeta}{dY} = \left(\frac{\rho_o}{\rho}\right) \frac{U}{u} - 1. \tag{2.13}$$

Thus we have reduced our problem to that of solving the following pair of first order ordinary differential equations

$$\frac{d}{d\overline{Y}} \left[\overline{A}^{-1/\gamma}\left(\frac{\overline{U}^2}{2} - \frac{\overline{u}^2}{2}\right)\right] - \frac{1}{F} \overline{\zeta} \frac{d\overline{A}^{-1/\gamma}}{d\overline{Y}} = 0 \tag{2.14}$$

$$\frac{d\overline{\zeta}}{d\overline{Y}} = \left(\frac{\overline{p}_o}{\overline{p}}\right)^{1/\gamma} \frac{\overline{U}}{\overline{u}} - 1 \tag{2.15}$$

where the non-dimensional barred variables are defined as $\overline{A} = A/A_g$, $\overline{U} = U/U_g$, $\overline{u} = u/U_g$, $\overline{y} = y/H$, $\overline{Y} = Y/H$, $\overline{p} = p/p_g$ and $\overline{p}_o = p_o/p_g$. Also p_o is defined as the pressure far upstream, H is the distance between the upper and lower surfaces far upstream, A_g, U_g and p_g are evaluated far upstream at Y=0 and F, the Froude number, is given by $F = U_g^2/gH$.

Using equation (2.10) we derive the following equation expressing the pressure in terms of velocity and the displacement of the streamline $\zeta(Y)$,

$$p(Y) = \left[p_o^{\frac{\gamma-1}{\gamma}} + \frac{F}{2k}(U^2-u^2) - \frac{1}{k}\zeta(Y)\right]^{\frac{\gamma}{\gamma-1}} \tag{2.16}$$

where $k = \frac{\gamma}{\gamma-1} \frac{p_g}{gH\rho_g}$. In general we determine $p_o(Y)$ using the hydrostatic relation (2.8) applied to the flow far upstream.

3. Analytic Solutions for Homentropic Flows.

Assuming that the flow is homentropic (i.e. constant entropy throughout the flow field), the solution of the system (2.14) and (2.15) can be represented analytically. The hydrostatic relation (2.8) applied to the flow far upstream of the obstacle takes the form

$$p_o^{-1/\gamma}\, dp_o = -gA^{-1/\gamma}\, dY$$

which, due to the assumption that $A(Y)$ = constant, can be integrated to yield

$$p_o(Y) = \left(1 - \frac{Y}{k}\right)^{\frac{\gamma}{\gamma-1}} . \tag{3.1}$$

Equations (2.14) and (2.15) become, using equations (2.16) and (3.1) and the assumption of homentropy

$$\frac{d}{dY}\,(U^2-u^2) = 0 \tag{3.2}$$

$$\frac{dy}{dY} = \left(\frac{k-Y}{\bar{k}-y}\right)^{\frac{1}{\gamma-1}} \frac{U}{u} \tag{3.3}$$

where $\bar{k} = k + \frac{1}{2}F(U^2-u^2)$.

The solution of (3.2)-(3.3) corresponding to a particular choice of $U(Y)$ and satisfying the rigid upper boundary condition $y(1) = 1$ is given by

$$y(Y) = \bar{k} - \left\{(\bar{k}-1)^{\frac{\gamma}{\gamma-1}} + \frac{\gamma}{\gamma-1}\int_Y^1 \frac{U(\hat{Y})}{[U(\hat{Y})^2+\alpha]^{1/2}}\,(k-\hat{Y})^{\frac{1}{\gamma-1}}\, d\hat{Y}\right\}^{\frac{\gamma-1}{\gamma}} \tag{3.4}$$

where we have used $u = [U^2(Y)+\alpha]^{1/2}$ and α is an integration constant of (3.2). The value of α is determined at any position downstream

by the obstacle height h immediately below this position through the equation

$$y(0) = h$$

and $\alpha=0$ corresponds to zero obstacle height. The allowable range of values of α is restricted b y the requirement that the pressure must remain positive. Using equation (2.16) we find that this implies that α must satisfy

$$\alpha < \frac{2}{F}(k-1) \tag{3.5}$$

which leads to the further restriction that k must be greater than 1 since $\alpha=0$ corresponds to unperturbed flow.

If we also assume that U(Y) is constant, the solution of (3.2)-(3.3) for the case of a rigid upper boundary becomes

$$y(Y) = \bar{k} - \{(\bar{k}-1)^{\frac{\gamma}{\gamma-1}} + \frac{U}{u}[(k-Y)^{\frac{\gamma}{\gamma-1}} - (k-1)^{\frac{\gamma}{\gamma-1}}]\}^{\frac{\gamma-1}{\gamma}} \ . \tag{3.6}$$

We define the critical Froude number F_c to be the Froude number for which the critical height is zero. It is easily shown that for the above solution (3.6)

$$F_c = \frac{\gamma-1}{\gamma}\left[\frac{k^{\frac{\gamma}{\gamma-1}} - (k-1)^{\frac{\gamma}{\gamma-1}}}{k^{\frac{1}{\gamma-1}} - (k-1)^{\frac{1}{\gamma-1}}}\right] . \tag{3.7}$$

In figures 3, 4 and 5 we plot the obstacle height y(0) versus u for the three qualitatively distinct cases $F < F_c$, $F = F_c$ and $F > F_c$. In figure 6 we plot the critical height h_c versus F.

It should be noted that there is no added difficulty in the case that the upper boundary is a free surface. The appropriate boundary condition for this case is that the pressure be constant along the uppermost streamline $Y=1$ or equivalently

$$y(1) = 1 + \frac{1}{2} F(U^2 - u^2) \qquad (3.8)$$

where we have used equation (2.16). The exact solution for this case, assuming homentropy and constant $U(Y)$, is

$$y(Y) = \bar{k} - \{(k-1)^{\frac{\gamma}{\gamma-1}} + \frac{U}{u}[(k-Y)^{\frac{\gamma}{\gamma-1}} - (k-1)^{\frac{\gamma}{\gamma-1}}]\}^{\frac{\gamma-1}{\gamma}} \qquad (3.9)$$

and the critical Froude number F_c is

$$F_c = \frac{\gamma-1}{\gamma}\left[\frac{k^{\frac{\gamma}{\gamma-1}} - (k-1)^{\frac{\gamma}{\gamma-1}}}{k^{\frac{1}{\gamma-1}}}\right]. \qquad (3.10)$$

We make the observation that our system of ordinary differential equations (2.14) and (2.15) approaches the equations governing incompressible flow over an obstacle as $\gamma \to \infty$. Taking this limit we note that $F_c \to \infty$ for the case of a rigid upper boundary and $F_c \to 1$ for the case of a free upper boundary.

The above results are easily extended to the case where entropy is constant in two or more horizontal layers which together make up the entire flow field.

4. Numerical Solutions for Nonhomentropic Flows.

For arbitrary velocity and entropy profiles far upstream of the obstacle we are no longer able to obtain analytic solutions. Therefore we integrate equations (2.14) and (2.15) numerically.

The procedure that was followed consisted of numerically integrating the initial value problem

$$\frac{d}{dY}\,[A^{-1/\gamma}(\frac{U^2}{2} - \frac{u^2}{2})] - \frac{1}{F}\,\zeta\,\frac{dA}{dY}^{-1/\gamma} = 0 \tag{2.14}$$

$$\frac{d\zeta}{dY} = (\frac{p_o}{p})^{1/\gamma}\,\frac{U}{u} - 1 \tag{2.15}$$

$$\zeta(1) = 0 \qquad u(1) = \alpha \tag{4.1}$$

from Y=1 to Y=0 using a 4th order Runge-Kutta-Gill subroutine. The parameter α is allowed to range over all positive numbers such that $p > 0$.

Frequently flows of this type contain stagnation points in the interior of the flow field. In order to perform the integration of equations (2.14)-(2.15) at positions on the downstream side of the stagnation point it is necessary to allow the streamline passing through the stagnation point to split (see figure 7).

Thus for positions on the downstream side of the stagnation point, the integration is done in two stages. First we integrate the initial value problem (2.14),(2.15) and (4.1) from Y=1 to $Y=Y_s$ where Y_s denotes the height far upstream of the streamline passing through the stagnation point. In order to complete the second stage of the integration (from $Y=Y_s$ to $Y=0$) we derive three jump conditions which constrain the flow in the region $Y < Y_s$. These jump conditions follow from the three constants of motion along streamlines (2.9),(2.10) and (2.11). From (2.9) we have that

$$[A] = 0 \tag{4.2}$$

where $[(\cdot)] = (\cdot)\Big|_{Y=Y_s^+} - (\cdot)\Big|_{Y=Y_s^-}$.

Equation (2.10), along with the hydrostatic assumption and the assumption that entropy is constant in any recirculating region bounded by a split streamline, implies

$$[u^2] = 0 \ . \tag{4.3}$$

The assumption of constant entropy in the recirculating region is motivated by stability considerations, although the present analysis cannot provide any details of the flow in this region. Equation (2.11) and the above results (4.2) and (4.3) imply

$$\frac{2}{F}\frac{\partial}{\partial Y}(\ell n\ A^{1/\gamma})[y] = [\frac{\partial}{\partial Y}(u^2)] \tag{4.4}$$

where $[y] = y(Y_s^+) - y(Y_s^-)$ is the splitting width of the split streamline, and is unspecified in this analysis.

This procedure was carried out for upstream velocity and entropy profiles corresponding to flow over the Rocky Mountains (see figures 8 and 9). Figure 10 is a plot of velocity u(1) along the uppermost streamline versus the obstacle height y(0). This is a representation of the master solution for this problem. Figure 11 is a plot of the streamlines for the flow over an obstacle whose height is equal to the critical height.

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List of Figures

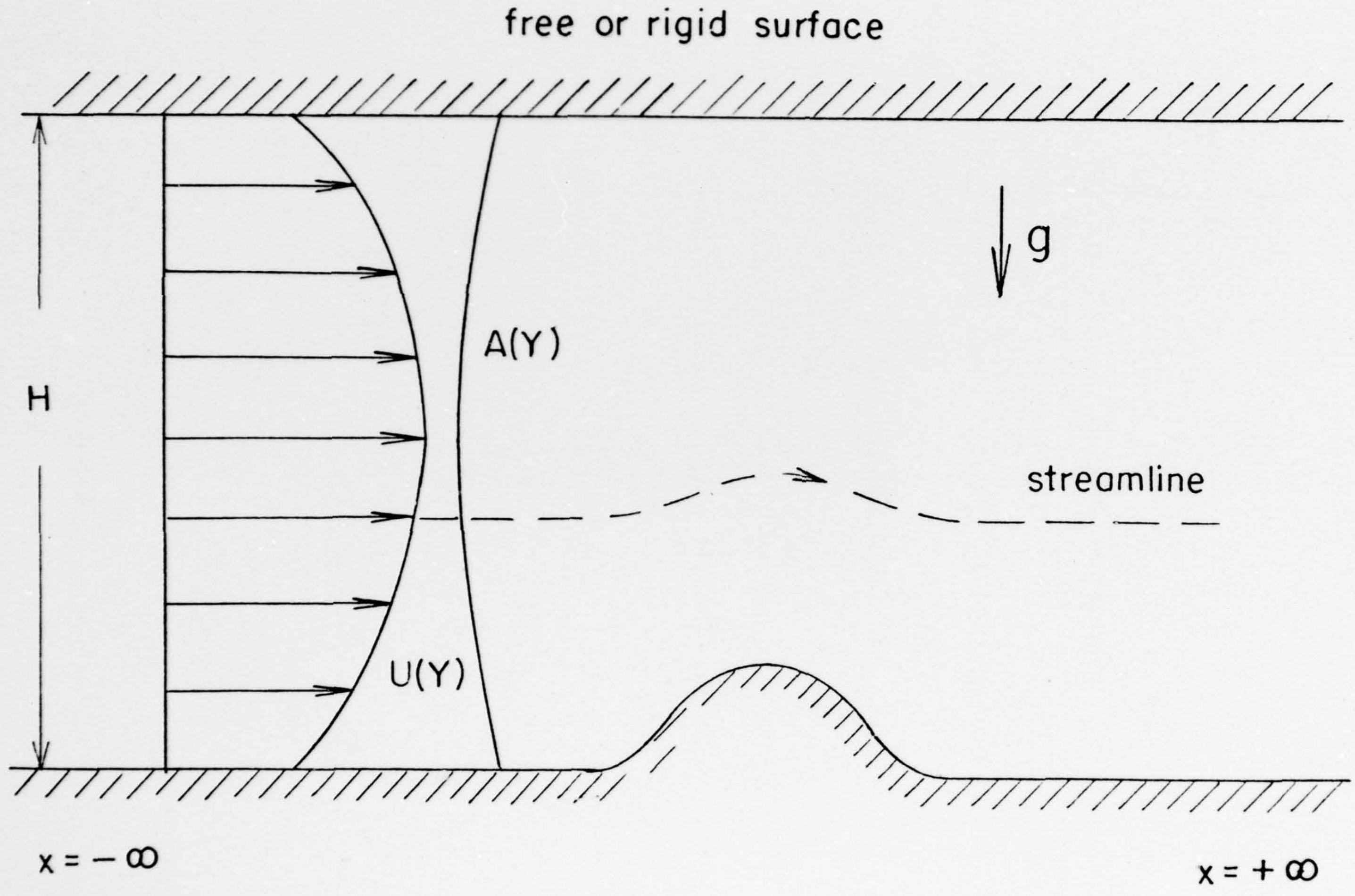

FIGURE 1

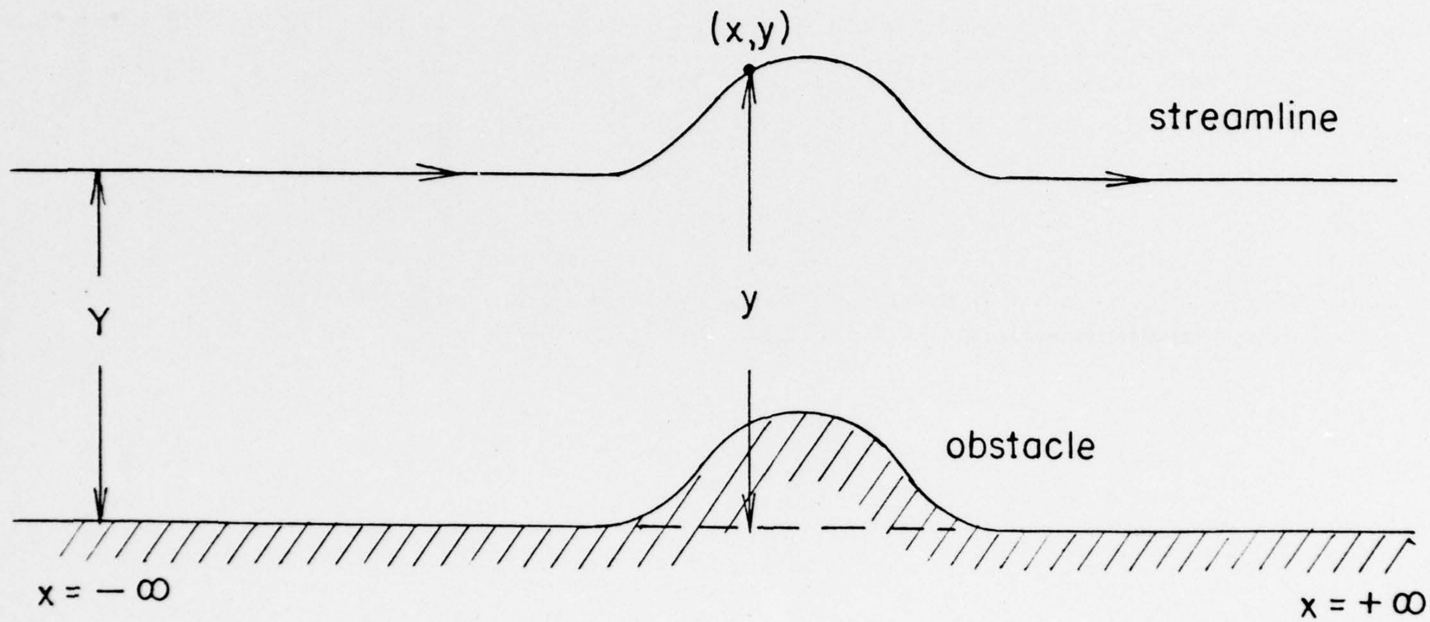

FIGURE 2

FIGURE 3

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FIGURE 4

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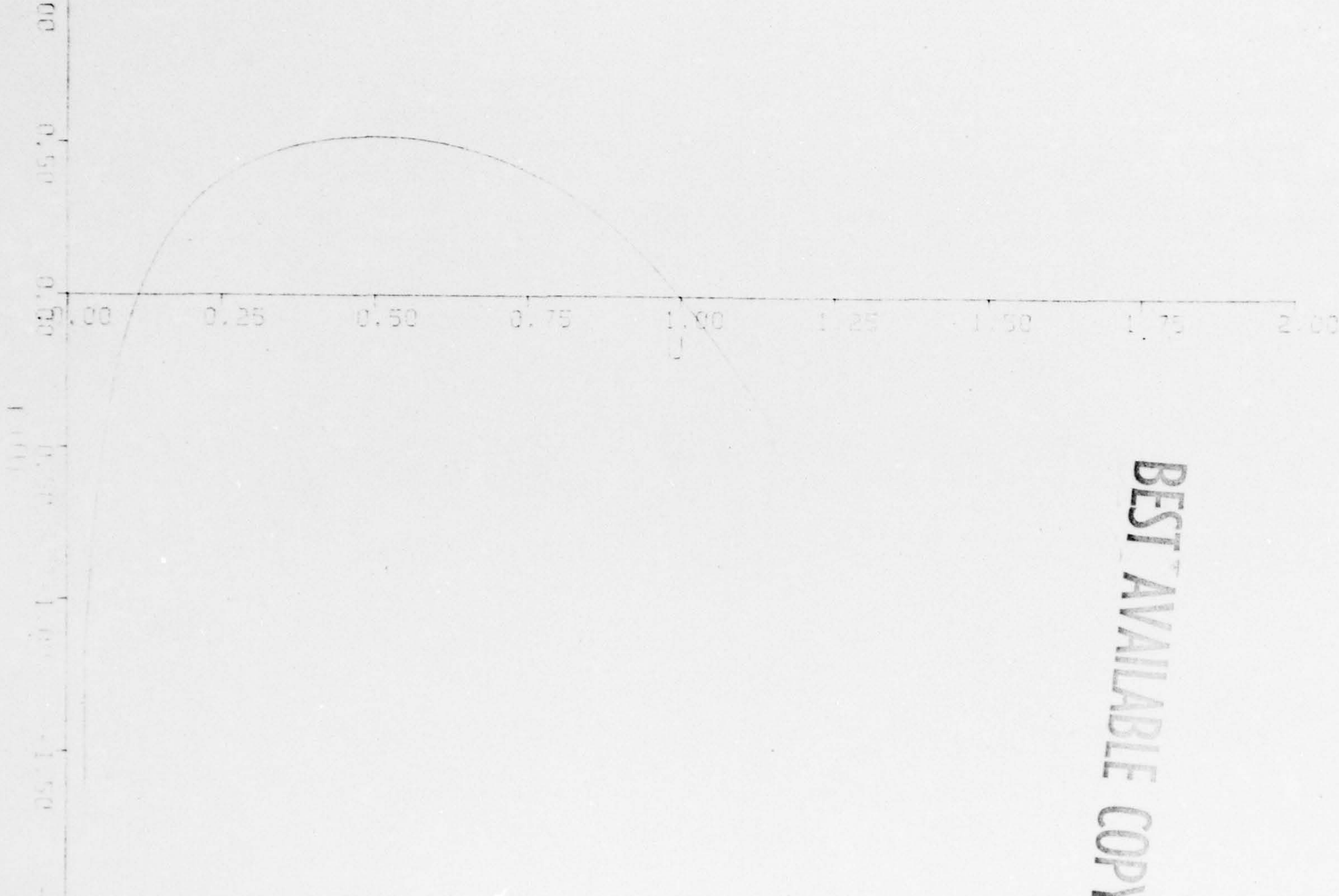

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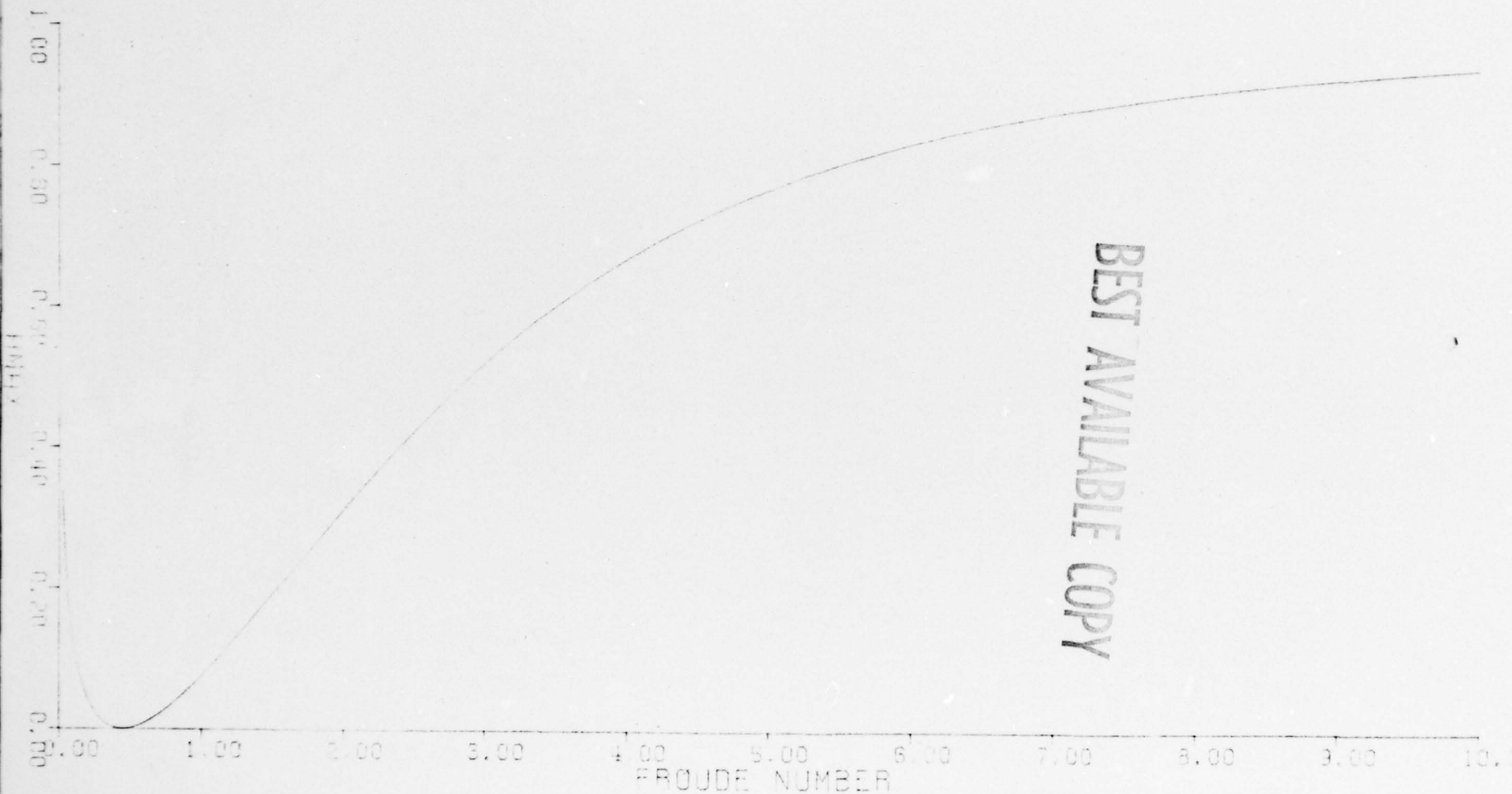

FIGURE 6

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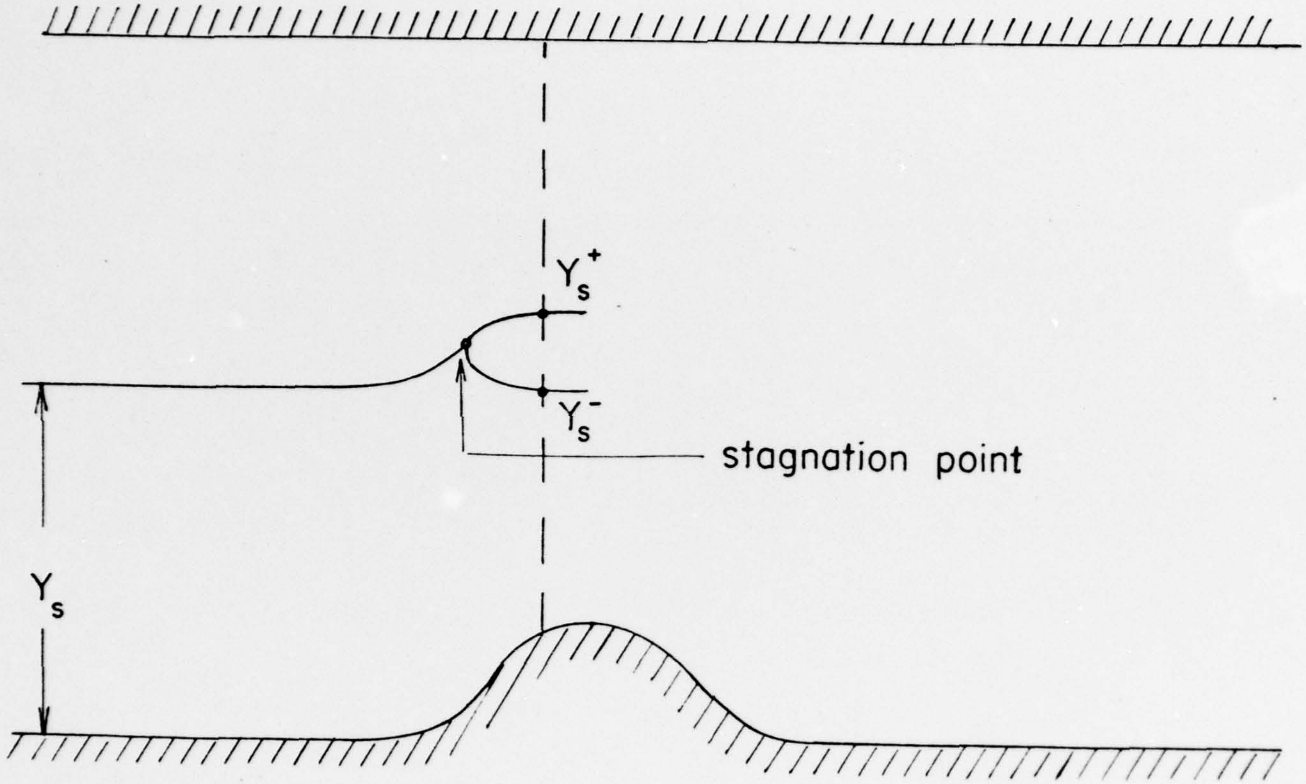

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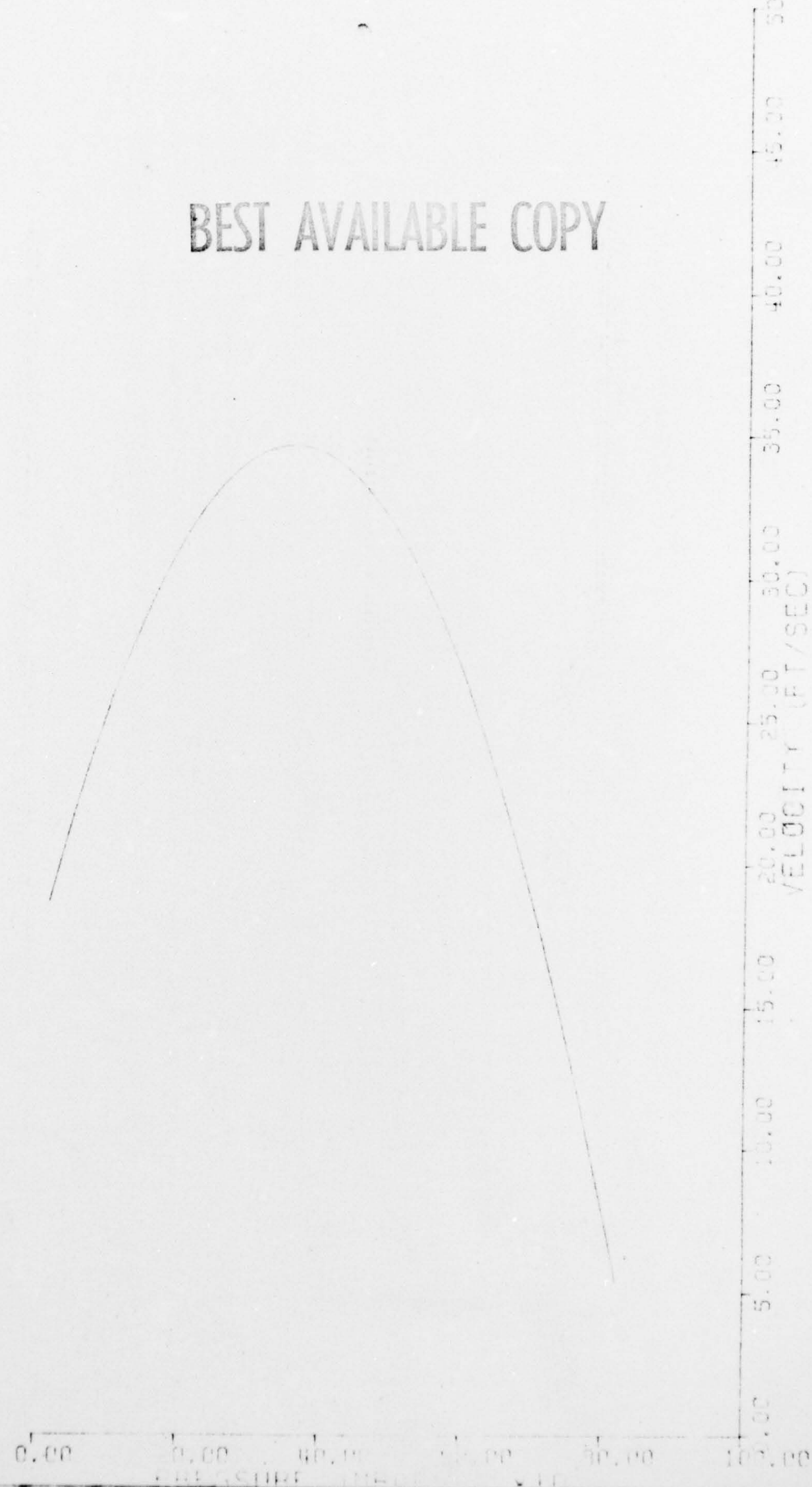

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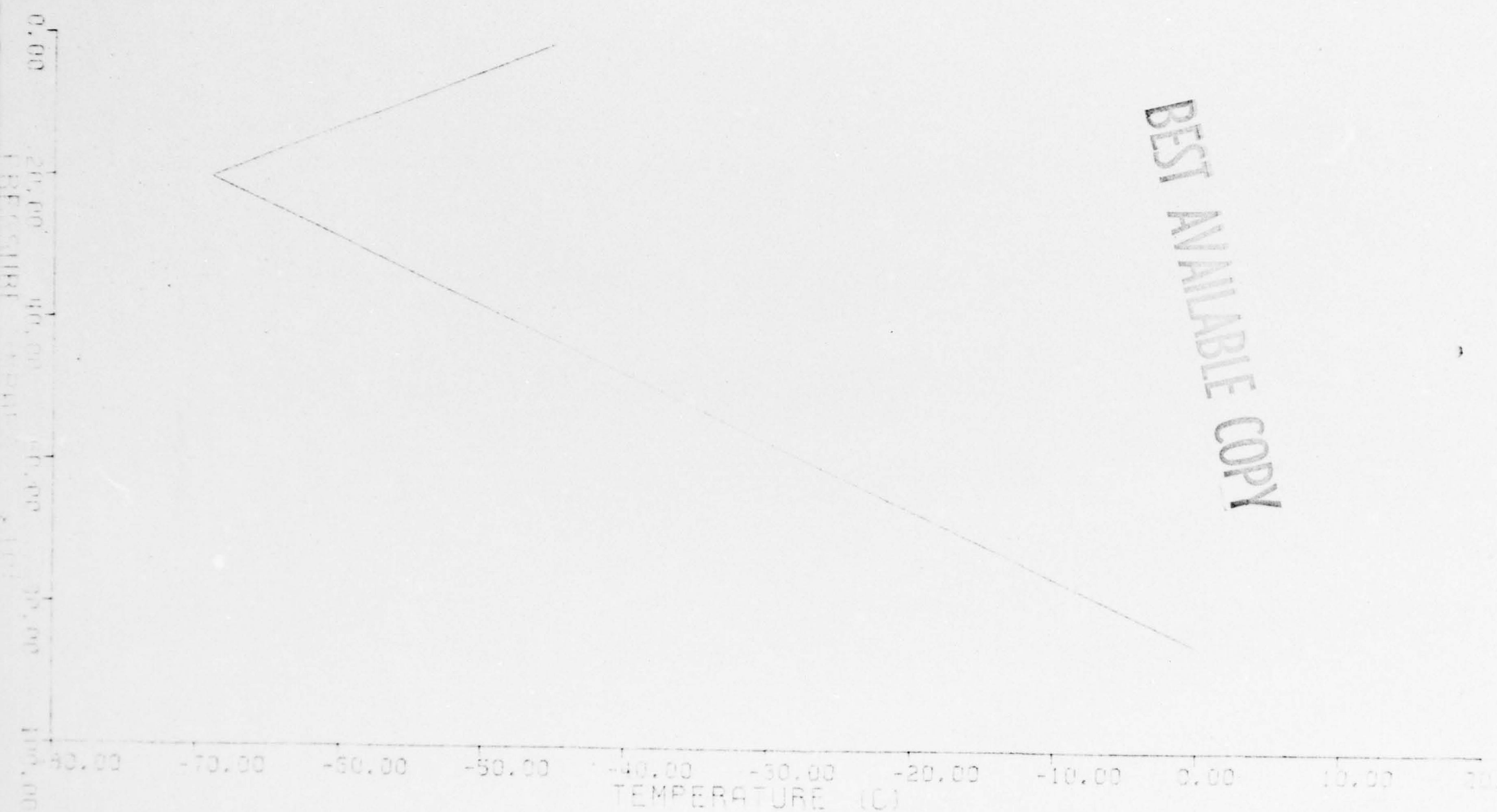

FIGURE 9

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FIGURE 10

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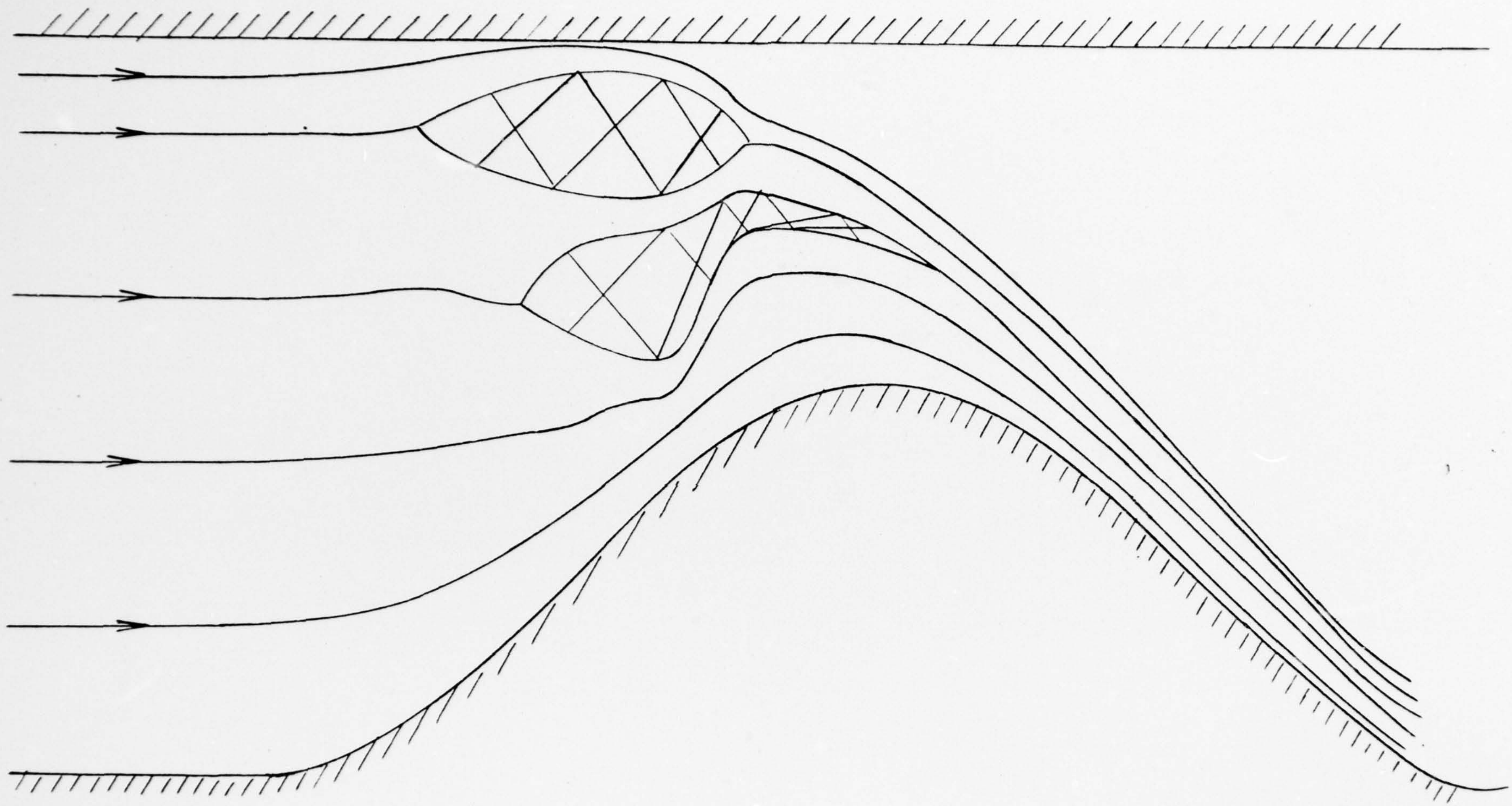

FIGURE II

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